THERMAL HISTORY (LECTURE 2)

	E	t	z
DARK MATTER FREEZEOUT	1 GeV?	10^{-8} s?	10^{13}?
NEUTRINO FREEZEOUT	1 MeV	1 s	10^{10}
BIG BANG NUCLEOSYNTHESIS	0.1 MeV	3 MIN	10^{8}
MATTER-RADIATION EQUALITY	1 eV	10^4 YR	10^4
RECOMBINATION	0.1 eV	10^5 YR	10^3
FIRST STARS	150 K	3×10^7 YR	60
REIONIZATION	25 K	4×10^8 YR	10
MATTER-C.C. EQUALITY	4 K	10^{10} YR	0.4
NOW	2.726 K	13.8 GYR	0

CONSIDER A SPECIES OF PARTICLE WHOSE OCCUPATION NUMBER IN EACH (MOMENTUM, SPIN) STATE IS A RANDOM VARIABLE. E.G. A PARTICLE IN THERMAL EQUILIBRIUM SUCH AS PHOTONS WITH A BLACKBODY DISTRIBUTION (WE WILL ALSO CONSIDER NON-THERMAL DISTRIBUTIONS). LET $f(p,t)$ BE THE MEAN OCCUPATION NUMBER. FOR A THERMAL DISTRIBUTION WITH TEMPERATURE T, f IS GIVEN BY

$$f = \begin{cases} \dfrac{\sum_{N=0}^{\infty} N e^{-NE/T}}{\sum_{N=0}^{\infty} e^{-NE/T}} & \text{BOSON} \\[2ex] \dfrac{\sum_{N=0}^{1} N e^{-NE/T}}{\sum_{N=0}^{1} e^{-NE/T}} & \text{FERMION} \end{cases}$$

$$= \frac{1}{e^{E/T} \mp 1}$$

WHERE THROUGHOUT THE LECTURE
UPPER SIGN = BOSON
LOWER SIGN = FERMION

A MODE SUM OVER (MOMENTUM, SPIN) STATES CAN BE APPROXIMATED

$$\sum_{\text{MODES}} \longrightarrow gV \int \frac{d^3p}{(2\pi)^3}$$

WHERE V = VOLUME

g = # OF SPIN STATES (E.G. $g=2$ FOR PHOTON)

EXAMPLE: ENERGY DENSITY OF THERMAL PARTICLE

$$\rho = \frac{1}{V} \sum_{\text{MODES}} E f$$

$$= g \int \frac{d^3p}{(2\pi)^3} E \frac{1}{e^{E/T} \mp 1} \qquad \text{WHERE } E = \sqrt{p^2 + m^2}$$

~~A SIMPLE CALCULATION SHOWS THAT~~

THIS INTEGRAL CAN'T BE DONE ANALYTICALLY, BUT DOES HAVE ANALYTIC FORMS IN THE RELATIVISTIC ($T \gg m$) AND NONRELATIVISTIC ($T \ll m$) LIMITS:

$$\rho \xrightarrow{T \gg m} \frac{\pi^2}{30} g_* T^4 \qquad \text{WHERE } g_* = \begin{cases} g & \text{BOSON} \\ \frac{7}{8} g & \text{FERMION} \end{cases}$$

$$\rho \xrightarrow{T \ll m} g m \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

[DETAILS OF CALCULATION LEFT AS EXERCISE]

GENERALIZES BLACKBODY CASE: $\rho = \frac{\pi^2}{15} T^4$ FOR PHOTON (MASSLESS BOSON WITH $g=2$)

LIKEWISE THE NUMBER DENSITY AND PRESSURE OF A THERMAL SPECIES CAN BE WRITTEN AS p-INTEGRALS AND SIMPLIFIED IN THE RELATIVISTIC AND NONRELATIVISTIC LIMITS

$$n = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{E/T} \mp 1}$$

$$\xrightarrow{T \gg m} 0.122\, g_{**} T^3 \qquad g_{**} = \begin{cases} g & \text{BOSON} \\ \frac{3}{4} g & \text{FERMION} \end{cases}$$

$$\xrightarrow{T \ll m} g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T}$$

$$P = g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} \frac{1}{e^{E/T} \mp 1}$$

$$\xrightarrow{T \gg m} \frac{1}{3}\rho$$ [RELATIVISTIC PARTICLE HAS $w = \frac{1}{3}$]

$$\xrightarrow{T \ll m} nT$$ [NON-REL PARTICLE OBEYS IDEAL GAS LAW]

WE NEXT DISCUSS THE TIME EVOLUTION OF $f(p,t)$

FIRST CONSIDER NON-INTERACTING CASE: WE HAVE A BATH OF PARTICLES WITH OCCUPATION NUMBER $f(p,t_1)$ AND ALL PARTICLES FREESTREAM TO LATER TIME t_2. WHAT IS $f(p,t_2)$?

WE SHOWED IN LECTURE 1 THAT FRW GEODESICS SATISFY $p \propto 1/a$, WHETHER TIMELIKE OR LIGHTLIKE

THEREFORE $f(p,t_2) = f\left(p \frac{a(t_2)}{a(t_1)}, t_1\right)$ [FREESTREAMING CASE]

EQUIVALENTLY:

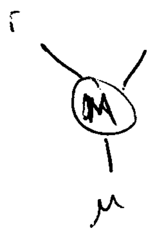
- $f = f(pa)$ IS A FUNCTION OF $pa(t)$ ALONE
- $\left(\frac{\partial}{\partial t} - H \frac{\partial}{\partial \log p} \right) f = 0$

$\hookrightarrow$ "LOUIVILLE OPERATOR"

GENERALIZATION TO INTERACTING (NON-FREESTREAMING) CASE: BOLTZMANN EQUATION

$$\left(\frac{\partial}{\partial t} - H \frac{\partial}{\partial \log p} \right) f = (\text{COLLISION TERM } C[f])$$
$$= (\text{EMISSION} - \text{ABSORPTION})$$

EXAMPLE: SUPPOSE f IS THE PHOTON DISTRIBUTION AND WE HAVE AN INTERACTION $\gamma + \Sigma_i \longleftrightarrow \Sigma_\mu$



EMISSION

ABSORPTION

$$\left(\frac{\partial}{\partial t} - H \frac{\partial}{\partial \log p} \right) f_\gamma(p,t) = \int \left(\prod_i \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right) \left(\prod_\mu \frac{d^3 p_\mu}{(2\pi)^3} \frac{1}{2E_{p_\mu}} \right)$$
$$\times (2\pi)^3 \delta^3\left(p + \Sigma p_i - \Sigma p_\mu\right)$$
$$\times (2\pi)\, \delta\left(E_p + \Sigma E_{p_i} - \Sigma E_{p_\mu}\right)$$
$$\times |M(p, p_i, p_\mu)|^2$$
$$\times \Big[(1+f_\gamma) \prod_i (1 \pm f_i) \prod_\mu f_\mu$$
$$- \prod_\mu (1 \pm f_\mu) \prod_i f_i\, f_\gamma \Big]$$

WHERE $\mathcal{M}$: LORENTZ INVARIANT AMPLITUDE
$1+f$ = BOSE-EINSTEIN ENHANCEMENT TERM
$1-f$ = PAULI BLOCKING TERM

NOTE: IT'S NOT HARD TO SHOW THAT IF ALL SPECIES ARE THERMAL WITH COMMON TEMPERATURE T, ~~THEN~~ (I.E. $f = (e^{E/T} \mp 1)^{-1}$) THEN THE RHS OF THE BOLTZMANN EQUATION IS ZERO FOR ANY SET OF INTERACTIONS WHICH CONSERVE ENERGY AND ARE TIME-REVERSAL INVARIANT (OR CP-INVARIANT).

IN GENERAL THE BOLTZMANN EQ CAN BE INTERPRETED AS A COMPETITION BETWEEN FREESTREAMING (LHS) AND THERMALIZATION (RHS)

FREESTREAMING WINS IF $\Gamma \ll H$
THERMALIZATION WINS IF $\Gamma \gg H$

WHERE $\Gamma = \left(\frac{\#\ \text{INTERACTIONS}}{\text{PARTICLE} \times \text{PROPER TIME}} \right)$

IN THE EARLY UNIVERSE, Γ GROWS FASTER WITH TEMPERATURE THAN H ($H \propto T^2$ ~~WHEREAS~~ WHEREAS $\Gamma \propto$ AT LEAST T^3)

SO PARTICLES START OUT IN THERMAL EQUILIBRIUM AND "FREEZE OUT" WHEN THEIR INTERACTION RATE DROPS BELOW THE HUBBLE RATE (FREESTREAMING THEREAFTER)

E.G. NEUTRINO FREEZEOUT OCCURS WHEN

$$\Gamma \sim H$$

$$n\sigma \sim H$$

$$T^3 (G_F T^2) \sim T^2 / M_{pl} \qquad \text{WHERE } G_F \sim 10^{-5}\ \text{GeV}^{-2} \ (\nu\bar{\nu} \leftrightarrow e^+e^-)$$

$$T \sim G_F^{-2/3} M_{pl}^{-1/3} \sim 1\ \text{MeV}$$

THEREAFTER THE NEUTRINO DISTRIBUTION FREESTREAMS

$$f(p,t) = \frac{1}{\exp\left(p \frac{a(t)}{a_{\text{freezeout}}} \Big/ T_{\text{freezeout}}\right) + 1}$$

NOT A THERMAL DISTRIBUTION IF $m_\nu > 0$!
(HAVE p IN "EXP" NOT E_p)

BUT IT IS CONVENTIONAL TO DEFINE A NEUTRINO "TEMPERATURE" $T_\nu = T_{\text{freezeout}}\, a_{\text{freezeout}} / a(t)$

SO THAT

$$f(p,t) = \frac{1}{e^{p/T_\nu} + 1} \qquad \left(\text{NOT } \frac{1}{e^{E/T_\nu} + 1} \ !\right)$$

NOTE THAT $T_\nu \propto 1/a$ BY DEFINITION

AFTER NEUTRINO FREEZEOUT, THE NEUTRINOS DECOUPLE FROM THE $e^+e^-\gamma$ THERMAL PLASMA

NEXT EVENT: e^+e^- PAIRS ANNIHILATE TO PHOTONS, LEAVING BEHIND A PLASMA OF PHOTONS + A FEW LEFTOVER e^- (BARYON ASYMMETRY)

WHAT HAPPENS TO TEMPERATURE T_γ DURING THIS ANNIHILATION EVENT? THERE ARE SEVERAL WAYS TO CALCULATE IT BUT THE EASIEST IS TO USE ENTROPY CONSERVATION. BECAUSE THE ANNIHILATION PROCESS TAKES PLACE IN THERMAL EQUILIBRIUM (ADIABATIC) ENTROPY IS CONSERVED

$$S = \frac{V(\rho + P)}{T}$$

WHY IS THIS THE CORRECT DEFINITION? BECAUSE FIRST LAW $d(\rho V) = T\,dS - p\,dV$ IS SATISFIED (EXERCISE) WHICH IMPLIES $dS = 0$ FOR ADIABATIC PROCESS

BEFORE e^+e^- ANNIHILATION

$$S = \frac{V}{T}\left(\sum \frac{\pi^2}{30} g_* T^4 + \frac{1}{3}\sum \frac{\pi^2}{30} g_* T^4\right)$$

$$= \frac{2\pi^2}{45}\left(\sum g_*\right) T^3 V$$

$$= \frac{2\pi^2}{45}\Big(\underbrace{2\cdot\tfrac{7}{8}}_{e^-} + \underbrace{2\cdot\tfrac{7}{8}}_{e^+} + \underbrace{2}_{\gamma}\Big) T^3 V$$

AFTER e^+e^- ANNIHILATION

$$S = \frac{2\pi^2}{45} \left(\Sigma g_*\right) T^3 V$$

$$= \frac{2\pi^2}{45} (2) T^3 V$$

SETTING ENTROPIES EQUAL AND NOTING $V \propto 1/a^3$

$$\left(2 \cdot \frac{7}{8} + 2 \cdot \frac{7}{8} + 2\right) T_{before}^3 \, a_{before}^{-3} = 2 \, T_{after}^3 \, a_{after}^{-3}$$

I.E.

$$T_{after} = \left(\frac{11}{4}\right)^{1/3} T_{before} \left(\frac{a_{before}}{a_{after}}\right)$$

WHICH IMPLIES THAT $T_\gamma \propto 1/a$, EXCEPT FOR A TRANSIENT BOOST BY $\left(\frac{11}{4}\right)^{1/3}$ DURING e^+e^- ANNIHILATION

SINCE $T_\nu \propto 1/a$ ALWAYS, THIS MEANS THAT

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

FOR ALL TIMES AFTER e^+e^- ANNIHILATION

$\Rightarrow$ PREDICT EXISTENCE OF COSMIC NEUTRINO BACKGROUND, WITH $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_{CMB} = 1.95$ K

NEUTRINO RELIC ABUNDANCE

$$\rho_\nu = 2\int \frac{d^3p}{(2\pi)^3}\sqrt{p^2+m_\nu^2}\,\frac{1}{e^{p/T_\nu}+1}$$

WHERE $T_\nu = 1.95\,K$
$= 0.17\,meV$

$$= \begin{cases} \frac{7\pi^2}{120}T_\nu^4 & \text{IF } m_\nu \ll T_\nu \\ \frac{2}{\pi^2}m_\nu T_\nu^3 & \text{IF } m_\nu \gg T_\nu \end{cases}$$

IN "OBSERVERS VARIABLES"

$$\Omega_\nu h^2 = \begin{cases} 5.4\times10^{-6} & m_\nu \ll T_\nu \\ \frac{m_\nu}{94\,eV} & m_\nu \gg T_\nu \end{cases}$$

THIS IS FOR A SINGLE NEUTRINO SPECIES

$(\Omega_\nu h^2)_{TOTAL}$ = SUM OVER ALL SPECIES

NOTE THAT $\sum m_\nu \geq 0.06$ eV BY NEUTRINO OSCILLATION EXPERIMENTS SO NONRELATIVISTIC APPROXIMATION

$\Omega_\nu h^2 = \frac{\sum m_\nu}{94\,eV}$ IS ALWAYS REASONABLY ACCURATE

CAN NEUTRINOS BE THE DARK MATTER? NO, FOR THE FOLLOWING REASON. ~~IF $\Omega_\nu h^2$~~ TO GET $\Omega_\nu h^2 \approx 0.1$ WE WOULD NEED $\sum m_\nu \approx 10$ eV WHICH IMPLIES A VELOCITY DISPERSION $\sigma_\nu \sim \frac{T_\nu}{m_\nu}$ $\sim 100\ km\ s^{-1}$ WHICH IS TOO LARGE ("HOT DARK MATTER") TO BE CONSISTENT WITH OBSERVATIONS

(FORMATION OF BOUND STRUCTURES WITH ROTATION VELOCITY $\lesssim 100$ KM S^{-1} WOULD BE SUPPRESSED, BUT SUCH STRUCTURES ARE OBSERVED)

SINCE THIS ARGUMENT DIDN'T USE ANY DETAILED PROPERTIES OF THE NEUTRINO, IT MAY APPEAR TO SHOW THAT THE DARK MATTER CAN'T BE A THERMAL RELIC. HOWEVER THERE IS A LOOPHOLE. THE ARGUMENT ACTUALLY APPLIES TO ANY SPECIES WHICH FREEZES OUT WHILE STILL RELATIVISTIC. A SPECIES WHICH GOES NONRELATIVISTIC ~~WHILE~~ BEFORE FREEZEOUT WILL HAVE ITS NUMBER DENSITY SUPPRESSED BY $e^{-M/T_{\text{freezeout}}}$. CAN GET ORDER-ONE $\Omega_c h^2$ BY COMBINING SMALL (RELATIVE TO CMB PHOTONS) NUMBER DENSITY AND LARGE MASS. THE LARGE MASS KEEPS THE VELOCITY DISPERSION SMALL ENOUGH TO SATISFY OBSERVATIONAL CONSTRAINTS. "COLD DARK MATTER"

DARK MATTER FREEZEOUT

MASS m_χ, COUPLING $\chi\chi \leftrightarrow$ SM w/ CROSS SECTION σ

CONDITION FOR COLD RELIC (STILL IN THERMAL EQUILIBRIUM WHEN $T \sim m_\chi$):

$$n\sigma \gg H \qquad \text{WHEN } T \sim m_\chi$$

$$T^3 \sigma \gg \frac{T^2}{M_{pl}}$$

$$\sigma \gg \frac{1}{T M_{pl}} = \frac{1}{m M_{pl}}$$

IF THIS IS SATISFIED, FREEZEOUT OCCURS IN NONRELATIVISTIC REGIME, WHEN

$$n\sigma \sim H$$

$$(m_\chi T)^{3/2} e^{-m_\chi/T} \sigma \sim \frac{T^2}{M_{pl}}$$

$$x^{1/2} e^{-x} \sim \frac{1}{\sigma m M_{pl}} \qquad \text{WHERE } x = m/T$$

EXPONENTIAL ON RHS MEANS $x \gg 1$ BUT NOT TOO LARGE (E.G. $\sigma \sim 10^{-8}$ GeV^{-2} AND $m \sim 1$ TeV GIVES $x \sim 30$)

DARK MATTER ABUNDANCE TODAY

$$\rho_{\text{TODAY}} \approx m_\chi n_{\text{freezeout}} \left(\frac{a_{\text{freezeout}}}{a_{\text{today}}} \right)^3$$

$$\sim m_\chi \left(\frac{H_{\text{freezeout}}}{\sigma} \right) \left(\frac{T_{CMB}}{T_{\text{freezeout}}} \right)^3$$

$$\sim m_\chi \left(\frac{T^2_{\text{freezeout}}}{\sigma M_{pl}} \right) \left(\frac{T_{CMB}}{T_{\text{freezeout}}} \right)^3$$

$$\sim \frac{x T^3_{CMB}}{\sigma M_{pl}}$$

IN "OBSERVERS VARIABLES"

$$\frac{\Omega_c}{0.2} \sim \left(\frac{x}{20} \right) \left(\frac{10^{-8} \text{ GeV}^{-2}}{\sigma} \right)$$

SO CRITERION FOR OBSERVED ABUNDANCE IS $\sigma \sim 10^{-8}$ GeV^{-2} WITH NO CONSTRAINT ON m_χ.

APPEARANCE OF ELECTROWEAK SCALE ($\sigma \sim 10^{-8}$ GeV^{-2}) IS SOMETIMES CALLED THE "WIMP MIRACLE"

SUGGESTS DM IS NEW STABLE PARTICLE WITH COUPLINGS TO THE STANDARD MODEL AT THE ELECTROWEAK SCALE (?)

NOTE RELIC ABUNDANCE IS INVERSELY PROPORTIONAL TO σ, SO WEAKEST COUPLED PARTICLE DOMINATES ("MEEK WILL INHERIT THE EARTH")

THE CANCELLATION OF m_χ IN $(\Omega_c h^2)$ MEANS THAT WE CAN'T INFER THE DARK MATTER PARTICLE MASS FROM COSMOLOGY, BUT THERE ARE SOME GENERAL ARGUMENTS WHICH BOUND IT:

1) IF COUPLING TO STANDARD MODEL SATISFIES $\sigma \sim G^2 m_\chi^2$ WITH $G \lesssim G_F$ TO AVOID NEW FORCES BELOW THE ELECTROWEAK SCALE WHICH HAVEN'T BEEN OBSERVED, THEN $\boxed{m_\chi \gtrsim (\text{few GeV})}$ [LEE-WEINBERG BOUND]

2) UNITARITY REQUIRES $\sigma \lesssim 4\pi m_\chi^{-2}$ WHICH IMPLIES $\boxed{m_\chi \lesssim 100 \text{ TeV}}$

DIRECT DM DETECTION EXPERIMENTS TYPICALLY CONCENTRATE ON THIS MASS RANGE