TRISEP problems: Electroweak Theory, SSB, and the Higgs

1. (a) Derive the transformations of ℓ_L, e_R, and ϕ in the standard model under an infinitesimal $SU(2) \times U(1)$ transformation by expanding $U = e^{i\left(\vec{\beta}(x)\cdot\vec{L}+\alpha(x)y\right)}$ to leading non-trivial order in $\vec{\beta}$ and α.
(b) Show explicitly that $\phi^\dagger\phi$, $\bar{\ell}_L\, i\not{D}\ell_L$, and $\bar{\ell}_L\phi\, e_R$ are invariant under these infinitestimal transformations.

2. Consider the standard model for a single fermion family and with $\mu^2 > 0$, i.e., no spontaneous symmetry breaking. Calculate the decay rate $\phi^0 \to \bar{d}d$, using the Yukawa interaction

$$-\mathcal{L}_{Yuk} = \Gamma^u \bar{q}_L \tilde{\phi} u_R + \Gamma^d \bar{q}_L \phi d_R + \Gamma^l \bar{\ell}_L \phi e_R + \Gamma^\nu \bar{\ell}_L \tilde{\phi} \nu_R + h.c.,$$

where (for one family) the Γ's are just numbers.

3. Consider a version of the $SU(2) \times U(1)$ theory with non-standard fermions. Instead of the three chiral families of quarks and leptons there is only a single non-chiral lepton doublet, with both the left and right-chiral fields transforming as $SU(2)$ doublets with $y = -\frac{1}{2}$, i.e.,

$$\ell_L = \begin{pmatrix} N \\ E^- \end{pmatrix}_L, \qquad \ell_R = \begin{pmatrix} N \\ E^- \end{pmatrix}_R.$$

Assume there is also a Higgs doublet ϕ, just as in the standard model.
(a) Write the Lagrangian density (before SSB) for $\ell_{L,R}$, including all allowed kinetic energy, gauge interaction, mass, and Yukawa terms.
(b) Now turn on SSB (i.e., take $\mu^2 < 0$), so that the mass eigenstate gauge bosons are $W^\pm$, Z, and A. Display the couplings of the leptons to $W^\pm$, Z, and A and to the Higgs scalar H.

4. (a) Show that (for two families) the observed quark masses and Cabibbo mixing can be approximately generated by

$$M^u = \frac{\Gamma^u \nu}{\sqrt{2}} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \qquad M^d = \frac{\Gamma^d \nu}{\sqrt{2}} = \begin{pmatrix} 0 & a \\ a & b \end{pmatrix},$$

where the parameters can be taken to be real and positive.
(b) Find numerical values for a and b.
Hint: even though M^d is Hermitian it may be necessary to choose $A^d_R \neq A^d_L$ to ensure positive eigenvalues.

5. Suppose that the standard model were extended by a fourth chiral family (t', b', e', ν') with masses (including that of the ν') $\gg m_t$. Estimate how the rates for $H \to ZZ^*$ and $H \to \gamma\gamma$ would be changed relative to the SM, assuming production by gluon fusion. Hint: assume that the widths for GG and $\gamma\gamma$ can be scaled from the equations given in the notes.

6. Consider a generalization of the $SU(2) \times U(1)$ model involving k multiplets $\phi_i, i = 1 \cdots k$, of complex scalars. The dimension of the i^{th} multiplet is $2t_i + 1$, where t_i can be $0, 1/2, 1, 3/2 \cdots$, and the elements have T^3 eigenvalues $t_i^3 = -t_i, -t_i + 1 \cdots t_i$ (cf., the rotation group). Also, the i^{th} multiplet has weak hypercharge y_i. Assume that each multiplet has one electrically neutral component ϕ_i^0, i.e., with $q_i = t_i^3 + y_i = 0$, and that that component acquires a vacuum expectation value $\langle \phi_i^0 \rangle = \nu_i/\sqrt{2}$.
(a) Show that the mass eigenstates $W^\pm$, Z, and A are the same as in the standard model.
(b) Calculate the W and Z masses in terms of g, g', t_i, t_i^3, and ν_i.
(c) The ρ_0 parameter, $\rho_0 \equiv M_W^2/(M_Z^2 \cos^2\theta_W)$ is predicted to be unity at the tree level in the standard model and in extensions involving additional Higgs doublets. Show that in the more general case

$$\rho_0 = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = \frac{\sum_{i=1}^k [t_i(t_i+1) - (t_i^3)^2] |\nu_i|^2}{2\sum_{i=1}^k (t_i^3)^2 |\nu_i|^2}.$$

(d) Specialize to the case of one doublet and two triplets

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi^{++} \\ \Phi^+ \\ \Phi^0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix},$$

where $\nu_\phi \equiv \sqrt{2}\langle \phi^0 \rangle \gg \nu_\Phi \equiv \sqrt{2}\langle \Phi^0 \rangle$ and $\nu_\phi \gg \nu_\Sigma \equiv \sqrt{2}\langle \Sigma^0 \rangle$. Calculate ρ_0 to leading nontrivial order in ν_Φ/ν_ϕ and ν_Σ/ν_ϕ.